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1984 J. Phys. A: Math. Gen. 17 851

(http://iopscience.iop.org/0305-4470/17/4/026)

J. Phys. A: Math. Gen. 17 (1984) 851–858. Printed in Great Britain

Exact solutions of the Dirac equation with a linear scalar confining potential in a uniform electric field

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Received 19 August 1983

Abstract. The exact solutions of the Dirac equation with a linear scalar confining potential in a uniform electric field are given. It is found that, if the scalar potential is stronger than that of the electric field, confinement is permanent. On the contrary, if the electric field is strong enough, confinement is impossible due to the Klein paradox.

1. Introduction

In the phenomenology of the quarkonium (Beavis *et al* 1979), the linear potential of the Schrödinger equation leads to the best fit of the J/ψ spectrum and the γ spectrum. However, when we extend this potential to the relativistic domain, as is pointed out by many authors (Gunion and Li 1975, Critchfield 1976), if the linear potential is vectorlike, the Dirac equation can not give the confining result since the Klein paradox exists. If we want to get a confining solution from the Dirac equation, we must introduce a scalarlike potential. A scalar potential in the Dirac equation is equivalent to a dependence of the rest mass upon position. Such potentials appear in the bag model (Chodos *et al* 1974) and other models of hadrons.

Recently, by means of the WKB method, Ni and Su (1980) and others (Fishbane *et al* 1983, Long and Robson 1983), have discussed the mixture of vectorlike and scalarlike potentials in Dirac equations and find, if the vectorlike potential is stronger than the scalarlike potential, confinement does not occur since the tunnelling solution arises. Tunnelling will be forbidden only if the confining potential of the Dirac scalar is stronger than the vectorlike potential. However, we must point out that all these methods are approximate since the exact solution of $(3+1)$-dimensional Dirac equation with linear scalar potential is, to our knowledge, still absent.

In this paper, after introducing a canonical transformation we can give the exact solutions of the Dirac equation with linear scalar potential Az which is only dependent on the direction z, as well as with a uniform electrical field which is a simple realisation of the Dirac vectorlike linear potential. Our results coincide with those obtained with the help of the approximate (WKB) method but, of course, our solutions are rigorously accurate.

2. The exact solutions of Dirac equation with scalarlike potential Az

The model considered by us consists of a Dirac scalarlike potential $V(z)=Az$ (A is a constant), the Dirac equation is

$$[\gamma^{\mu}\hbar\,\partial/\partial x^{\mu}+(mc+Az)]\psi(\boldsymbol{r},t)=0 \qquad (1)$$

0305-4470/84/040851+08\$02.25

where $\psi(\boldsymbol{r}, t) = (\psi_1, \psi_2, \psi_3, \psi_4)$ is a Dirac spinor wavefunction, γ^μ is a 4×4 Dirac matrix. Let

$$\psi(\boldsymbol{r}, t) = \exp[\mathrm{i}(p_1 x + p_2 y - Et)/\hbar]\varphi(z) \tag{2}$$

where $\varphi(z) = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$, we get

$$\gamma^3\hbar(\mathrm{d}/\mathrm{d}z)\varphi(z) = [(E/c)\gamma^4 - (mc + Az) - \mathrm{i}\gamma^1 p_1 - \mathrm{i}\gamma^2 p_2]\varphi(z). \tag{3}$$

Operating on both sides of equation (3) by $\gamma^3\hbar\,\mathrm{d}/\mathrm{d}z$, gives

$$\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{pmatrix} = \left[-E^2/c^2 + p_1^2 + p_2^2 + (Az+mc)^2 - \hbar A\begin{pmatrix}0 & 0 & -\mathrm{i} & 0\\ 0 & 0 & 0 & \mathrm{i}\\ \mathrm{i} & 0 & 0 & 0\\ 0 & -\mathrm{i} & 0 & 0\end{pmatrix}\right]\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{pmatrix}. \tag{4}$$

We can rewrite equation (4) as

$$\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\begin{pmatrix}\varphi_1\\ \varphi_3\end{pmatrix} = \left[-\frac{E^2}{c^2} + p_1^2 + p_2^2 + (Az+mc)^2 - \hbar A\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}\right]\begin{pmatrix}\varphi_1\\ \varphi_3\end{pmatrix} \tag{5}$$

$$\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\begin{pmatrix}\varphi_2\\ \varphi_4\end{pmatrix} = \left[-\frac{E^2}{c^2} + p_1^2 + p_2^2 + (Az+mc)^2 + \hbar A\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}\right]\begin{pmatrix}\varphi_2\\ \varphi_4\end{pmatrix}. \tag{6}$$

After introducing a canonical transformation

$$\begin{pmatrix}\varphi_1\\ \varphi_3\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ \mathrm{i} & -\mathrm{i}\end{pmatrix}\begin{pmatrix}U_1\\ U_2\end{pmatrix} \tag{7}$$

$$\begin{pmatrix}\varphi_2\\ \varphi_4\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ \mathrm{i} & -\mathrm{i}\end{pmatrix}\begin{pmatrix}\tilde{U}_1\\ \tilde{U}_2\end{pmatrix} \tag{8}$$

and setting

$$\xi = (A/\hbar)^{1/2}(z + mc/A) \tag{9}$$

equations (5) and (6) reduce to

$$(\mathrm{d}^2/\mathrm{d}\xi^2)f(\xi) + \{[(E^2 - c^2p_1^2 - c^2p_2^2)/A\hbar c] + 1 - \xi^2\}f(\xi) = 0 \tag{10}$$

$$(\mathrm{d}^2/\mathrm{d}\xi^2)g(\xi) + \{[(E^2 - c^2p_1^2 - c^2p_2^2)/A\hbar c] - 1 - \xi^2\}g(\xi) = 0 \tag{11}$$

where f is either U_1 or $\tilde{U}_2$ and g is either U_2 or $\tilde{U}_1$. Obviously, (10) and (11) are just the same as those of the harmonic oscillator Schrödinger equation. If we choose the boundary condition $z \to \infty$ $\psi \to 0$, we get the energy spectrum

$$E = \pm[2(n+1)A\hbar c + c^2p_1^2 + c^2p_2^2]^{1/2} \tag{12}$$

and

$$f(\xi) = H_{n+1}(\xi)\exp(-\tfrac{1}{2}\xi^2) \tag{13}$$

$$g(\xi) = H_n(\xi)\exp(-\tfrac{1}{2}\xi^2), \tag{14}$$

where $H(\xi)$ is the Hermite polynomial of degree n. Therefore

$$\begin{pmatrix}\varphi_1\\ \varphi_3\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ \mathrm{i} & -\mathrm{i}\end{pmatrix}\begin{pmatrix}C_1H_{n+1}(\xi)\\ C_2H_n(\xi)\end{pmatrix}\exp(-\tfrac{1}{2}\xi^2) \tag{15}$$

and

$$\begin{pmatrix}\varphi_2\\ \varphi_4\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ \mathrm{i} & -\mathrm{i}\end{pmatrix}\begin{pmatrix}C_3H_n(\xi)\\ C_4H_{n+1}(\xi)\end{pmatrix}\exp(-\tfrac{1}{2}\xi^2). \tag{16}$$

These are the solutions of equations (4), but we cannot guarantee that they must also be the solutions of equation (3), since we have operated on both sides of equation (3) by the operator $\gamma^3\hbar\, \mathrm{d}/\mathrm{d}z$. To avoid this ambiguity, let us substitute (15) and (16) into equation (3). After a short calculation, it can be shown that if we choose the coefficients C_1, C_2, C_3, C_4 satisfying

$$\begin{cases} 2(n+1)C_1-E(A\hbar c^2)^{-1/2}C_2+(p_2+\mathrm{i}p_1(A\hbar)^{-1/2}C_3=0\\ -(p_2-\mathrm{i}p_1)(A\hbar)^{-1/2}C_2+E(A\hbar c^2)^{-1/2}C_3-2(n+1)C_4=0\\ E(A\hbar c^2)^{-1/2}C_1-C_2+(p_2+\mathrm{i}p_1)(A\hbar)^{-1/2}C_4=0\\ -(p_2-\mathrm{i}p_1)(A\hbar)^{-1/2}C_1+C_3-E(A\hbar c^2)^{-1/2}C_4=0, \end{cases} \tag{17}$$

(15) and (16) must be the solutions of equation (3). It can easily be shown that the determinant of the coefficients of equation (17) is zero if the energy spectrum equation (12) is satisfied, and

$$C_2=E(A\hbar c^2)^{-1/2}C_1+(p_2+\mathrm{i}p_1)(A\hbar)^{-1/2}C_4 \tag{18}$$

$$C_3=(p_2-\mathrm{i}p_1)(A\hbar)^{-1/2}C_1+E(A\hbar c^2)^{-1/2}C_4. \tag{18'}$$

Finally, we get the exact solutions of the Dirac equation with linear scalar potential $V(z)=Az$ as

$$\psi_1=C_1\begin{pmatrix} H_{n+1}(\xi)+E(A\hbar c)^{-1/2}H_n(\xi)\\ (p_2-\mathrm{i}p_1)(A\hbar)^{-1/2}H_n(\xi)\\ \mathrm{i}H_{n+1}(\xi)-\mathrm{i}E(A\hbar c)^{-1/2}H_n(\xi)\\ -(p_2+\mathrm{i}p_1)(A\hbar)^{-1/2}H_n(\xi)\end{pmatrix}\exp(-\tfrac{1}{2}\xi^2)\exp[\mathrm{i}(p_1x+p_2y-Et)/\hbar] \tag{19}$$

$$\psi_2=C_4\begin{pmatrix} (p_2+\mathrm{i}p_1)(A\hbar)^{-1/2}H_n(\xi)\\ E(A\hbar c^2)^{-1/2}H_n(\xi)+H_{n+1}(\xi)\\ (p_1-\mathrm{i}p_2)(A\hbar)^{-1/2}H_n(\xi)\\ \mathrm{i}E(A\hbar c^2)^{-1/2}H_n(\xi)-\mathrm{i}H_{n+1}(\xi)\end{pmatrix}\exp(-\tfrac{1}{2}\xi^2)\exp[\mathrm{i}(p_1x+p_2y-Et)/\hbar]. \tag{20}$$

Obviously, if we choose the initial condition $p_1=p_2=0$, (19), (20) are confining bound states on the z direction, since V is only dependent on z.

3. The exact solutions combining with a uniform electric field

If there is a uniform electric field in the z direction, the Dirac equation with scalar potential $V(z)=Az$ will become

$$[\gamma^\mu(\hbar\,\partial/\partial x^\mu-(\mathrm{i}e/c)A_\mu)+(mc+Az)]\psi(\boldsymbol{r},t)=0 \tag{21}$$

where $A_\mu=-\mathrm{i}e\varepsilon z\delta_{\mu 4}$ is a four-dimensional Dirac electromagnetic potential and ε is the intensity of the electric field. In a similar way to that used in § 2 we get

$$\hbar^2 \frac{d^2}{dz^2}\begin{pmatrix}\varphi_1\\ \varphi_3\end{pmatrix} = \left[-\frac{(E+e\varepsilon z)^2}{c^2} + (Az+mc)^2 + p_1^2 + p_2^2 - i\frac{e\varepsilon\hbar}{c}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - A\hbar\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right]\begin{pmatrix}\varphi_1\\ \varphi_3\end{pmatrix}$$

$$\hbar^2 \frac{d^2}{dz^2}\begin{pmatrix}\varphi_2\\ \varphi_4\end{pmatrix} = \left[-\frac{(E+e\varepsilon z)^2}{c^2} + (Az+mc)^2 + p_1^2 + p_2^2 - i\frac{e\varepsilon\hbar}{c}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - A\hbar\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right]\begin{pmatrix}\varphi_2\\ \varphi_4\end{pmatrix}.$$

To solve equations (22) and (23), let us investigate the matrix

$$i\frac{e\varepsilon\hbar}{c}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - A\hbar\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{\hbar}{c}\begin{pmatrix}0 & i(e\varepsilon+Ac)\\ i(e\varepsilon-Ac) & 0\end{pmatrix} \tag{24}$$

its eigenvalue equation is

$$\begin{vmatrix}-\lambda & i(e\varepsilon+Ac)\\ i(e\varepsilon-Ac) & -\lambda\end{vmatrix} = 0 \tag{25}$$

and the eigenvalue $\lambda^2 = A^2c^2 - e^2\varepsilon^2$ may be real or imaginary. Let us discuss these two cases respectively.

(1) $A^2c^2 > e^2\varepsilon^2$

As the eigenvalues are real we can introduce the canonical transformation as follow

$$\begin{pmatrix}\varphi_1\\ \varphi_3\end{pmatrix} = \begin{pmatrix}(Ac+e\varepsilon)^{1/2} & (Ac+e\varepsilon)^{1/2}\\ -i(Ac-e\varepsilon)^{1/2} & i(Ac-e\varepsilon)^{1/2}\end{pmatrix}\begin{pmatrix}U_1'\\ U_2'\end{pmatrix} \tag{26}$$

$$\begin{pmatrix}\varphi_2\\ \varphi_4\end{pmatrix} = \begin{pmatrix}(Ac+e\varepsilon)^{1/2} & (Ac+e\varepsilon)^{1/2}\\ -i(Ac-e\varepsilon)^{1/2} & i(Ac-e\varepsilon)^{1/2}\end{pmatrix}\begin{pmatrix}\tilde{U}_1'\\ \tilde{U}_2'\end{pmatrix} \tag{27}$$

and setting

$$\xi' = \frac{(A^2c^2-e^2\varepsilon^2)^{1/4}}{(\hbar c)^{1/2}}\left[z + \frac{Amc^3 - e\varepsilon E}{A^2c^2-e^2\varepsilon^2}\right] \tag{28}$$

we have

$$(d^2/d\xi^2)f'(\xi') + [\alpha - 1 - \xi'^2]f'(\xi') = 0 \tag{29}$$

$$(d^2/d\xi^2)\,g'(\xi') + [\alpha + 1 - \xi^2]g'(\xi') = 0 \tag{30}$$

where f' is U_1' or $\tilde{U}_2'$, and g' is U_2' or $\tilde{U}_1'$, and

$$\alpha = \frac{c}{\hbar(A^2c^2-e^2\varepsilon^2)^{1/2}}\left[\frac{E^2}{c^2} - p_1^2 - p_2^2 - m^2c^2 + \frac{(Amc^3-e\varepsilon E)^2}{c^2(A^2c^2-e^2\varepsilon^2)}\right]. \tag{31}$$

Similarly, after some calculation we find the energy spectrum

$$E = \frac{mce\varepsilon}{A} \pm \left[2(n+1)A\hbar c\left(\frac{A^2c^2-e^2\varepsilon^2}{A^2c^2}\right)^{3/2} + (p_1^2c^2 + p_2^2c^2)\left(\frac{A^2c^2-e^2\varepsilon^2}{A^2c^2}\right)\right] \tag{32}$$

and wavefunctions

$$\psi^1(\boldsymbol{r},\,t) = C_2 \begin{pmatrix} (Ac-e\varepsilon)^{1/2}\alpha_1 H_n(\xi') + (Ac+e\varepsilon)^{1/2} H_{n+1}(\xi) \\ -\beta_2(Ac+e\varepsilon)H_n(\xi') \\ \mathrm{i}(Ac+e\varepsilon)^{1/2}\alpha_1 H_n(\xi') + \mathrm{i}(Ac-e\varepsilon)^{1/2} H_{n+1}(\xi') \\ -\mathrm{i}\beta_2(Ac-e\varepsilon)^{1/2} H_n(\xi') \end{pmatrix}$$

$$\times \exp(-\tfrac{1}{2}\xi'^2)\exp[\mathrm{i}(p_1x+p_2y-Et)/\hbar] \tag{33}$$

$$\psi^2(\boldsymbol{r},\,t) = C_3 \begin{pmatrix} -\beta_1(Ac+e\varepsilon)^{1/2} H_n(\xi') \\ (Ac+e\xi)^{1/2} H_{n+1}(\xi') + \alpha_1(Ac-e\xi)^{1/2} H_n(\xi') \\ \mathrm{i}\beta_1(Ac-e\varepsilon)^{1/2} H_n(\xi') \\ -\mathrm{i}(Ac-e\varepsilon)^{1/2} H_{n+1}(\xi') - \mathrm{i}\alpha_2(Ac+e\varepsilon)H_n(\xi') \end{pmatrix}$$

$$\times \exp(-\tfrac{1}{2}\xi'^2)\exp[\mathrm{i}(p_1x+p_2y-Et)/\hbar] \tag{34}$$

where

$$\begin{cases} \alpha_1 = \dfrac{1}{(\hbar c)^{1/2}(A^2c^2-e^2\varepsilon^2)^{1/4}}\left(E+mc^2-\dfrac{Amc^3-e\varepsilon E}{Ac-e\varepsilon}\right) \\ \alpha_2 = \dfrac{1}{(\hbar c)^{1/2}(A^2c^2-e^2\varepsilon^2)^{1/4}}\left(-E+mc^2-\dfrac{Amc^3-e\varepsilon E}{Ac+e\varepsilon}\right) \\ \beta_1 = -\left(\dfrac{c}{\hbar}\right)^{1/2}\dfrac{p_2+\mathrm{i}p_1}{(A^2c^2-e^2\varepsilon^2)^{1/4}} \\ \beta_2 = -\left(\dfrac{c}{\hbar}\right)^{1/2}\dfrac{p_2-\mathrm{i}p_1}{(A^2c^2-e^2\varepsilon^2)^{1/4}}. \end{cases} \tag{35}$$

If we put $p_1=p_2=0$ at $t=0$, ψ^1, ψ^2 are the respective bound states. It means that if the scalar potential is sufficiently strong compared to the electric field, the confinement will occur.

(2) $A^2c^2<e^2\varepsilon^2$.

As the eigenvalues are imaginary, we can introduce the transformation as

$$\begin{pmatrix}\varphi_1\\ \varphi_3\end{pmatrix} = \begin{pmatrix}(e\varepsilon+Ac)^{1/2} & (e\varepsilon+Ac)^{1/2} \\ (e\varepsilon-Ac)^{1/2} & -(e\varepsilon-Ac)^{1/2}\end{pmatrix}\begin{pmatrix}V_1\\ V_2\end{pmatrix} \tag{36}$$

$$\begin{pmatrix}\varphi_2\\ \varphi_4\end{pmatrix} = \begin{pmatrix}(e\varepsilon+Ac)^{1/2} & (e\varepsilon+Ac)^{1/2} \\ (e\varepsilon-Ac)^{1/2} & -(e\varepsilon-Ac)^{1/2}\end{pmatrix}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\end{pmatrix} \tag{37}$$

and change the variable as

$$\eta = \frac{(e^2\varepsilon^2-A^2c^2)^{1/4}}{(\hbar c)^{1/2}}\left[z-\frac{Amc^3-e\varepsilon E}{e^2\varepsilon^2-A^2c^2}\right] \tag{38}$$

then V_1 or $\tilde{V}_2$ satisfies

$$(\mathrm{d}^2/\mathrm{d}\eta^2)\tilde{f}(\eta)+(\eta^2+\beta-\mathrm{i})\tilde{f}(\eta)=0 \tag{39}$$

V_2 or $\tilde{V}_1$ satisfies

$$(\mathrm{d}^2/\mathrm{d}\eta^2)\tilde{g}(\eta)+(\eta^2+\beta+\mathrm{i})\tilde{g}(\eta)=0 \tag{40}$$

where

$$\beta=\frac{c}{\hbar(e^2\varepsilon^2-A^2c^2)^{1/2}}[E^2/c^2-m^2c^2-p_1^2-p_2^2-(Amc^3-e\varepsilon E)^2/(c^2(e^2\varepsilon^2-A^2c^2)]. \tag{41}$$

The solutions of equations (40) and (41) are parabolic cylinder functions (Gradshteyn and Ryzhik 1980)

$$\tilde{f}(\eta)=D_{\beta i/2}(\pm(1-i)\eta) \tag{42}$$

$$\tilde{g}(\eta)=D_{\beta(i-1)/2}(\pm(1-i)\eta). \tag{43}$$

Similarly, we get the exact solution of equation (21) as

$$\psi^1(\boldsymbol{r},t)=C_1\begin{pmatrix}(e\varepsilon+Ac)^{1/2}D_\rho(\omega)-\tilde{\alpha}_1(e\varepsilon-Ac)^{1/2}D_{\rho-1}(\omega)\\ -(e\varepsilon+Ac)^{1/2}\tilde{\beta}_2D_{\rho-1}(\omega)\\ (e\varepsilon-Ac)^{1/2}D_\rho(\omega)-\tilde{\alpha}_2(e\varepsilon+Ac)^{1/2}D_{\rho-1}(\omega)\\ -(e\varepsilon-Ac)^{1/2}\tilde{\beta}_2D_{\rho-1}(\omega)\end{pmatrix}\exp[i(p_1x+p_2y-Et)/\hbar] \tag{44}$$

$$\psi^2(\boldsymbol{r},t)=C_4\begin{pmatrix}-\tilde{\beta}_2(e\varepsilon+Ac)^{1/2}D_{\rho-1}(\omega)\\ -\tilde{\alpha}_1(e\varepsilon-Ac)^{1/2}D_{\rho-1}(\omega)+(e\varepsilon+Ac)^{1/2}D_\rho(\omega)\\ \tilde{\beta}_2(e\varepsilon-Ac)^{1/2}D_{\rho-1}(\omega)\\ \tilde{\alpha}_2(e\varepsilon+Ac)^{1/2}D_{\rho-1}(\omega)-(e\varepsilon-Ac)^{1/2}D_\rho(\omega)\end{pmatrix}\exp[i(p_1x+p_2y-Et)/\hbar] \tag{45}$$

where

$$\begin{cases}\tilde{\alpha}_1=\frac{1}{2}(1-i)\dfrac{1}{(\hbar c)^{1/2}(e^2\varepsilon^2-A^2c^2)^{1/2}}\left[\dfrac{Amc^3-e\varepsilon E}{e\varepsilon-Ac}+E+mc^2\right]\\ \tilde{\alpha}_2=\frac{1}{2}(1-i)\dfrac{1}{(\hbar c)^{1/2}(e^2\varepsilon^2-A^2c^2)^{1/2}}\left[\dfrac{Amc^3-e\varepsilon E}{e\varepsilon+Ac}+E-mc^2\right]\\ \tilde{\beta}_1=-\frac{1}{2}(1+i)\left(\dfrac{c}{\hbar}\right)^{1/2}\dfrac{(p_2+ip_1)}{(e^2\varepsilon^2-A^2c^2)^{1/2}}\\ \tilde{\beta}_2=-\frac{1}{2}(1+i)\left(\dfrac{c}{\hbar}\right)^{1/2}\dfrac{(p_2-ip_1)}{(e^2\varepsilon^2-A^2c^2)^{1/2}}\end{cases} \tag{46}$$

$$\omega=-(1-i)\eta \qquad \rho=\tfrac{1}{2}\beta i \tag{47}$$

$$\psi^3(\boldsymbol{r},t)=C_1'\begin{pmatrix}(e\varepsilon+Ac)^{1/2}D_\rho(\omega')-\tilde{\alpha}_1'(e\varepsilon-Ac)^{1/2}D_{\rho-1}(\omega')\\ -(e\varepsilon+Ac)^{1/2}\tilde{\beta}_2'D_{\rho-1}(\omega')\\ (e\varepsilon-Ac)^{1/2}D_\rho(\omega')-\tilde{\alpha}_2'(e\varepsilon+Ac)^{1/2}D_\rho(\omega')\\ -(e\varepsilon-Ac)^{1/2}\tilde{\beta}_2'D_{\rho-1}(\omega')\end{pmatrix}$$
$$\times\exp[i(p_1x+p_2y-Et)/\hbar] \tag{48}$$

$$\psi^4(\boldsymbol{r},t)=C_4'\begin{pmatrix}-\tilde{\beta}_2'(e\varepsilon+Ac)^{1/2}D_{\rho-1}(\omega')\\ -\tilde{\alpha}_1'(e\varepsilon-Ac)^{1/2}D_{\rho-1}(\omega')+(e\varepsilon+Ac)^{1/2}D_\rho(\omega')\\ \tilde{\beta}_2'(e\varepsilon-Ac)^{1/2}D_{\rho-1}(\omega')\\ \tilde{\alpha}_2'(e\varepsilon+Ac)^{1/2}D_\rho(\omega')-(e\varepsilon-Ac)^{1/2}D_\rho(\omega')\end{pmatrix}$$
$$\times\exp[i(p_1x+p_2y-Et)/\hbar] \tag{49}$$

where

$$\tilde{\alpha}_1' = -\tilde{\alpha}_1 \qquad \tilde{\alpha}_2' = -\tilde{\alpha}_2 \qquad \tilde{\beta}_1' = -\tilde{\beta}_1 \qquad \tilde{\beta}_2' = -\tilde{\beta}_2 \qquad \omega' = -\omega \tag{50}$$

and the corresponding energy spectrum is continuous. Then asymptotically we have when $z \to -\infty$

$$\psi^1(\boldsymbol{r}, t) \to C_1 \begin{pmatrix} (e\varepsilon + Ac)^{1/2} \\ 0 \\ (e\varepsilon - Ac)^{1/2} \\ 0 \end{pmatrix} \exp(\tfrac{1}{8}\pi\beta) \times \exp\{i[\tfrac{1}{2}\eta^2 + \tfrac{1}{2}\beta^2 \ln(\sqrt{2}|\eta|)]\} \exp[i(p_1 x + p_2 y - Et)/\hbar] \tag{51}$$

when $z \to +\infty$

$$\begin{aligned} \psi^1(\boldsymbol{r}, t) \to {} & C_1 \begin{pmatrix} (e\varepsilon + Ac)^{1/2} \\ 0 \\ (e\varepsilon - Ac)^{1/2} \\ 0 \end{pmatrix} \exp(-\tfrac{3}{8}\pi\beta) \\ & \times \exp\{i[\tfrac{1}{2}\eta^2 + \tfrac{1}{2}\beta^2 \ln(\sqrt{2}\eta)]\} \exp[i(p_1 x + p_2 y - Et)/\hbar] \\ & - C_1 \begin{pmatrix} \tilde{\alpha}_1 (e\varepsilon - Ac)^{1/2} \\ \tilde{\beta}_2 (e\varepsilon + Ac)^{1/2} \\ \tilde{\alpha}_2 (e\varepsilon + Ac)^{1/2} \\ \tilde{\beta}_2 (e\varepsilon - Ac)^{1/2} \end{pmatrix} \frac{\sqrt{2\pi}\exp(-\frac{1}{8}\pi\beta)}{\Gamma(-\frac{1}{2}\beta i + 1)} \\ & \times \exp\{i[\tfrac{1}{2}\eta^2 + \tfrac{1}{2}\beta^2 \ln(\sqrt{2}\eta)]\} \exp[i(p_1 x + p_2 y - Et)/2] \end{aligned} \tag{52}$$

and when $z \to -\infty$

$$\begin{aligned} \psi^3(\boldsymbol{r}, t) \to {} & C_1' \begin{pmatrix} (e\varepsilon + Ac)^{1/2} \\ 0 \\ (e\varepsilon - Ac)^{1/2} \\ 0 \end{pmatrix} \exp(-\tfrac{3}{8}\pi\beta) \\ & \times \exp\{i\tfrac{1}{2}\eta^2 + \tfrac{1}{2}\beta \ln(\sqrt{2}|\eta|)]\} \exp[i(p_1 x + p_2 y - Et)/\hbar] \\ & - C_1' \begin{pmatrix} \tilde{\alpha}_1 (e\varepsilon - Ac)^{1/2} \\ \tilde{\beta}_2' (e\varepsilon + Ac)^{1/2} \\ \tilde{\alpha}_2' (e\varepsilon + Ac)^{1/2} \\ \tilde{\beta}_2' (e\varepsilon - Ac)^{1/2} \end{pmatrix} \frac{\sqrt{2\pi}\exp(-\frac{1}{8}\pi\beta)}{\Gamma(-\frac{1}{2}\beta i + 1)} \\ & \times \exp i[\tfrac{1}{2}\eta^2 + \tfrac{1}{2}\beta \ln(\sqrt{2}|\eta|)] \exp[i(p_1 x + p_2 y - Et)/\hbar] \end{aligned} \tag{53}$$

when $z \to +\infty$

$$\psi^3(\boldsymbol{r}, t) \to C_1' \begin{pmatrix} (e\varepsilon + Ac)^{1/2} \\ 0 \\ (e\varepsilon - Ac)^{1/2} \\ 0 \end{pmatrix} \exp(\tfrac{1}{8}\pi\beta) \exp i[\tfrac{1}{2}\eta^2 + \tfrac{1}{2}\beta \ln(\sqrt{2}\eta)] \times \exp[i(p_1 x + p_2 y - Et)/\hbar] \tag{54}$$

where Γ is the gamma function. The asymptotic behaviour of ψ^2 and ψ^4 is similar. They represent the incident waves, reflected waves and transmitted waves. Then the confinement is not existed even if $p_1 = p_2 = 0$ at $t = 0$.

4. Summary and discussion

We have seen that the exact solution for the linear scalarlike potential $V(z) = Az$ Dirac equation in the presence of a uniform electric field are parabolic cylinder functions (if $e^2\varepsilon^2 > A^2c^2$) or Hermite polynomials bound states (if $e^2\varepsilon^2 < A^2c^2$). The former is the tunnelling solution and the famous Klein paradox exists. The latter corresponds to the confined solutions in the z direction. This conclusion is in agreement with the approximate result obtained by Ni and Su (1980), Fishbane *et al* (1983), but our conclusion is rigorous.

As a working hypothesis, in this paper, we imagine the linear scalarlike potential is only dependent on one dimension, say z, then we get z direction confinement only.

Finally, we must point out that if we take the limit $A \to 0$ solutions (43), (44), (47), (48) will reduce to the solutions of Ley Koo *et al* (1983), and Ni and Sullivan (1982), who investigated the behaviour of the Dirac equation in a uniform of electric field.

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